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Shape and Size Matter for Projectile Drag

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Cover Page Footnote

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I. INTRODUCTION

Modeling projectile motion using basic Newtonian mechanics is a standard activity in high school and first-year college physics classes. Such experiments may involve tracking the projectile motion of a certain object, for example a thrown ball, and then using Newton's Second Law to calculate the value of g, the acceleration due to gravity. However, frequently students cite that there is some air resistance that they have not taken into account that leads to an inaccurate measurement. Here our question is whether it is possible to fully model projectile motion taking this air drag into account. To do this, we will create a theoretical model including drag and also test the veracity of the model experimentally.

In doing this, we will also be able to answer a more specific question, namely whether drag varies with the shape and/or size of the projectile. In the context of an analytical mechanics course at the junior level, these turn out to be questions which are now readily accessible for investigation, as students are now equipped with the requisite knowledge in physics and differential equations.

In this paper, we report the results of such an investigation. First, we derive a theoretical model for the motion of a projectile moving through the air using Newton's laws, taking into account air resistance. Then we record videos of student experimenters throwing different projectiles using a high-speed camera. We analyze these videos using the software program Tracker [1], which allows us to track the motion of the projectile in every frame, and also plot the trajectory of the projectile which is predicted from the theoretical model. Next, we adjust the various parameters in our initial model until we obtain the best possible fit between model and data. We perform these steps for two projectiles of different shapes and sizes: a spherical ping pong ball and a cylindrical earplug. This enables us to compare the effect of drag between different sizes and shapes of projectiles. We find that shape and size do matter in determining the amount of drag a projectile experiences.

II. THEORETICAL BACKGROUND

To derive our equations of motion for a projectile traveling through the air, we start with Newton's Second Law, which states

$$\mathbf{F} = m\ddot{\mathbf{r}}, \tag{1}$$

where $\mathbf{F}$ is the sum of all the forces acting on the object, m is its mass, and $\mathbf{r}$ is the displacement of the projectile. The dots above the $\mathbf{r}$ indicate the second time derivative. We can break the displacement vector into its components:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}, \tag{2}$$

where $\hat{\mathbf{x}}$ is parallel to the ground and directed away from the experimenter, and $\hat{\mathbf{y}}$ is perpendicular to the ground and directed upwards (see Fig. 1). Thus we only consider motion in two dimensions. For our model, the forces that are not negligible are that of gravity and drag. The force of gravity ($\mathbf{F}_g$) acts downwards in the $-\hat{\mathbf{y}}$ direction, and the drag force ($\mathbf{F_d}$) acts opposite the velocity of the projectile. Hence Eq. 1 becomes

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_g + \mathbf{F}_d \\ &= -mg\hat{\mathbf{y}} - f(v)\hat{\mathbf{v}}, \end{aligned} \tag{3}$$

where m is the mass of the projectile, $g = 9.81$ m/s^2 is the acceleration due to gravity on Earth, $f(v)$ is a function characterizing the drag force and $\hat{\mathbf{v}}$ is the unit vector in the direction of the velocity. To finish the model we only need to characterize the drag force by finding $f(v)$.

In general, any function can be approximated by taking its Taylor expansion. If we do this for $f(v)$, we obtain

$$f(v) = bv + cv^2 + ... \tag{4}$$

where $v = |\mathbf{v}|$ is the speed of the object, and b and c are constants characterizing the strength of the linear and quadratic terms, respectively. Depending on the conditions of the projectile, it is not always necessary to take into account both terms. Which of the terms is relevant can be determined by the dimensionless Reynolds number R, which for a moving sphere in a fluid is defined as

$$R = \frac{Dv\rho}{\eta}, \tag{5}$$

where D is the diameter of the sphere, ρ the density of the fluid, and η is the viscosity of the fluid [2]. A high Reynolds number ($R > 1000$) indicates only quadratic drag matters, while a low number ($R < 1$) leaves us with only linear drag to take into account [3]. In between these extremes, both terms can be pertinent to the problem. For all of our projectiles, we can make a rough calculation of the Reynolds number by estimating $D \approx 5.0$ cm, $v \approx 1$ m/s (from a student lightly lobbing up the projectile into the air), $\rho = 1$ kg/m^3 for air and $\eta = 10^{-5}$ Pa·s for air. From this we obtain $R \approx 10^4$, which means that only quadratic drag needs to be considered in our model. Hence we use only the quadratic term in Eq. 4, and our revised force equation is

$$\mathbf{F} = m\ddot{\mathbf{r}} = -mg\hat{\mathbf{y}} - cv^2\hat{\mathbf{v}}. \tag{6}$$

To simulate this theoretical model in the Tracker program, we need to separate the components of $\mathbf{F}$ in the x- and y- directions. The velocity vector can be separated into the x- and y- directions by using the definition of the two unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{\sqrt{v_x^2 + v_y^2}} = \frac{v_x\hat{\mathbf{x}} + v_y.\hat{\mathbf{y}}}{v} \tag{7}$$

The force can also be broken up into its components,

$$\mathbf{F} = F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}}, \tag{8}$$

yielding our equations of motion in the x- and y- directions:

$$m\ddot{r}_x = -cv_x\sqrt{v_x^2 + v_y^2} \tag{9}$$

$$m\ddot{r}_y = -mg - cv_y\sqrt{v_x^2 + v_y^2}. \tag{10}$$

In order to calculate the position of the projectile, one only needs to integrate these equations. Due to the coupling between the equations, a simple analytical solution is not available, and thus one needs to do a numerical integration. This will be carried out by Tracker.

Finally, while we will be adjusting the parameter c until we obtain the best fit of model to the data, one can also calculate the parameter by using the relation

$$c = \kappa\rho A, \tag{11}$$

where κ depends on the shape of the projectile, ρ is the density of the fluid and A is the cross-sectional area of the projectile normal to the velocity [2]. It will be an interesting point to compare the predicted values of c based on Eq. 11 to the actual values we get by adjusting the model. Using this equation, we calculate our expected values for c for both projectiles (Tab. I). The listed error is determined by standard techniques of error propagation [4]. We expect that the ping pong ball will have a higher c than the ear plug as it is bigger. However, since $\kappa = 0.24$ for a sphere and 0.41 for a cylinder [5], we expect c for the ear plug to be larger than a spherical projectile of similar dimensions. As the cylinder undergoes forward rotation about an axis perpendicular to its length during its flight, the cross-sectional area in Eq. 11 is effectively rectangular, and so to calculate it we use the expression

$$A_{cylinder} = l \times d, \tag{12}$$

where l is the length and d is the diameter of the cylinder. To find the expected c for a spherical projectile of similar dimensions, we use the same $A_{cylinder}$ but multiplied by κ for a sphere. This corresponds to a sphere with $d \approx 1.8$ cm. Thus we see that as our projectiles differ in shape and size, we are able to investigate both of these effects.

Projectile	Shape	Size (cm)	Mass (g)	Expected c (kg m^{-1})
Ping pong ball	sphere	d=3.77(1)	2.6(1)	$2.7(1) \times 10^{-4}$
Earplug	cyllinder	d=1.11(1) l=2.38(1)	0.46(1)	$1.08(1) \times 10^{-4}$
Sphere of similar dimensions as earplug	sphere	$d \approx 1.8$	-	$6.34(8) \times 10^{-5}$

TABLE I. Table of projectile sizes. Note that d is diameter and l is length.

III. EXPERIMENTAL METHOD

Using a high-speed camera (Nikon 1 V1, 400 frames per second), we film the trajectories of projectiles thrown by student experimenters (Fig. 1). A dark background and lighting in the form of three studio lights are used in order to achieve clarity. One experimenter stands in the background holding up a meter-long stick in the middle (ensuring the ends are clearly visible in the video), which is useful for calibration purposes. The other experimenter gently lobs up the projectile at $\sim 45°$ angle and the camera films the progress of the projectile as it traces an approximately parabolic path, rising and then falling back to the ground mainly due to gravity. The object is thrown such as to minimize projectile spin as much as possible. If the projectile spins, it will be subject to the Magnus effect [6], which is not taken into account in our model. Care is also taken so that the camera is filming at roughly the same height as the initial height of the projectile as it leaves the experimenter's hand. Otherwise, the coordinate axes will be tilted at angle towards the camera and cause error when we are analyzing the motion in the x- vs. y-directions.

We then use the Tracker program developed by Doug Brown at Cabrillo College [1] to track the motion of the projectile. The flight of the projectile takes about a second or less from launch to falling back to the ground. This corresponds to 300-400 frames captured by the camera that show the projectile in mid-flight. Using Tracker, we go through these frames 5 frames at a time, marking by hand the position of the projectile's center of mass in each observed frame. This results in a track of 50-60 points, which depict the motion of the object. As this method of tracking relies mainly on the experimenter's visual judgment, one might expect it to be inaccurate. However, due to the small size of the projectile on the screen, we estimate that the method is good to within a 5% error in the position, based on the projectile's progress from frame to frame. Using the meter stick held in the background, we calibrate the program such that we record actual distances. Once the projectile's trajectory has been

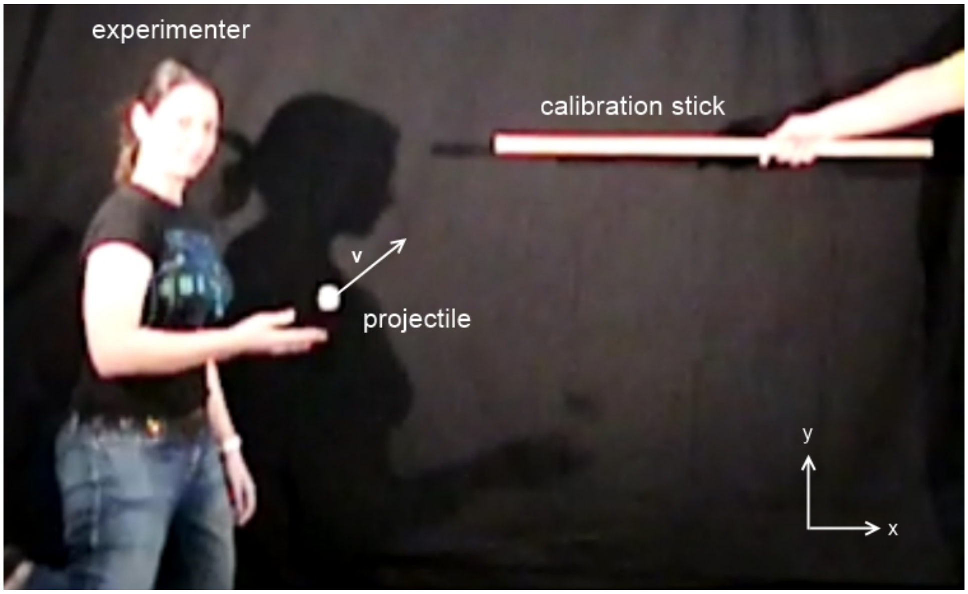

FIG. 1. View of experiment from camera. The experimenter lobs the projectile with an initial velocity v, such that the ball moves in the x-y plane defined by the coordinate axes. A meter stick placed in the frame allows the computer program to calibrate length. The camera is placed a distance of ~ 2 m from the dark background screen.

defined by the user, Tracker outputs the x(t) and y(t) positions along with the velocity and acceleration of the projectile throughout the flight.

Once we have plotted the data, we input our theoretical model into the program so as to compare our model with the data. We input Eq. 9 and Eq. 10 along with the initial conditions of the projectile. The program then integrates these equations of motion and plots the theoretical trajectory of the projectile. Next we adjust the value of c in order to match our theoretical plots of x-position versus time and y-position versus time with the actual trajectory: for each plot, we find a c_{max} and c_{min} such that the actual trajectory is contained within the area between the plots of c_{max} and c_{min} as much as possible (Fig. 2). Next, we plot the graph of y-position versus x-position (Fig. 3) and adjust c until the theoretical plot lies on top of the actual trajectory as closely as possible. This c becomes our experimental value, while the larger of the two c_{max} and the smaller of the two c_{min} determine our uncertainty. This analysis is done for both projectiles.

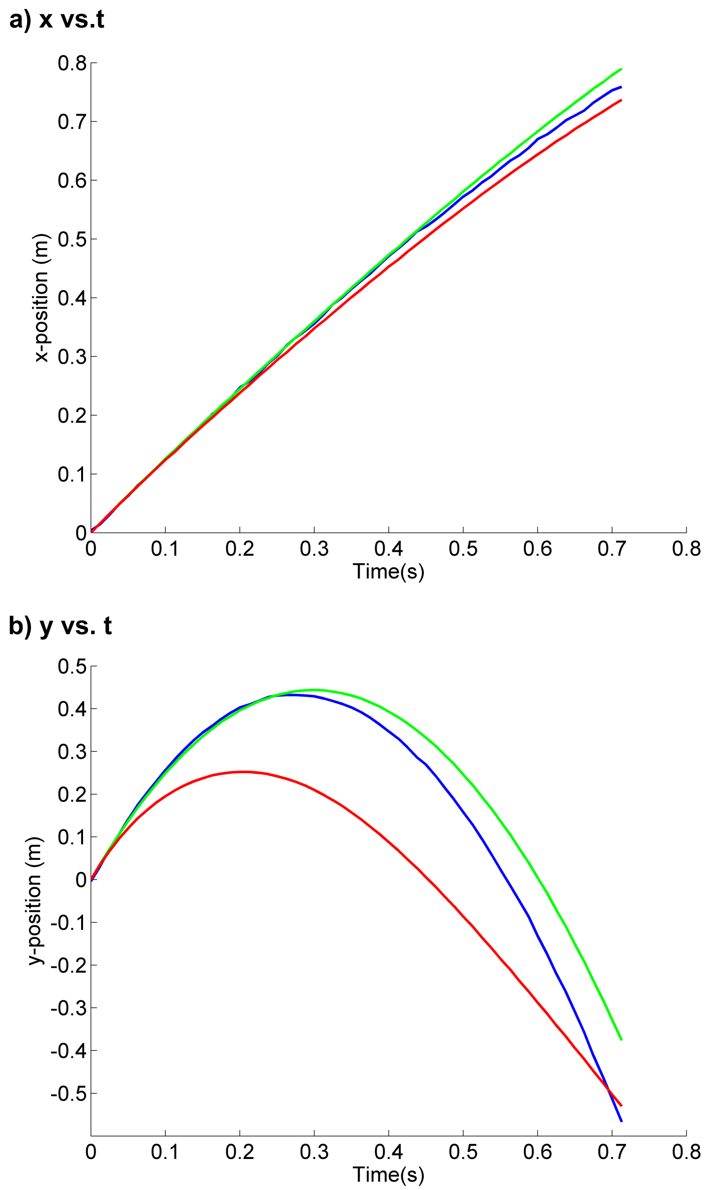

FIG. 2. Method of fitting curve, here shown for the ping pong ball. For both a) x vs. t and b) y vs. t, we find values of c_{max} (red line) and c_{min} (green line) such that as much of the actual trajectory (blue) is contained in the are between the two.

IV. RESULTS

Here, we present the results of our experiment. Without taking drag into account, the projectiles trace a perfectly parabolic trajectory through the air, as we have often assumed in introductory physics courses. With drag, we expect the the projectile's trajectory to be altered. Since velocity in both x- and y-directions is slowed down by the drag, the projectile should cover less distance in both coordinates. We expect the amount by which the trajectory is thus affected to depend on the projectile's shape and size: in other words, based on values of c which we had calculated earlier.

It turns out that this is indeed what we obtain, as seen Fig. 3. Both projectiles exhibit trajectories which are under the ideal parabolic trajectory (black line), in line with our expectations. However, we see that uncertainty in the trajectories is quite large, as it is difficult to find a value of c which matches both the x vs. t and y vs t. plots. In addition, we can also see the error bars in the actual trajectory which is from the 5% error estimate we made earlier. Despite these concerns, in general the theoretical trajectories do fit the actual trajectories well, much better than the ideal parabolic trajectories. Thus taking drag into account improves our understanding of projectile motion.

Projectile	Expected c (kg m^{-1})	Obtained c (kg m^{-1})
Ping pong ball	$2.7(1) \times 10^{-4}$	$8^{+42}_{-6} \times 10^{-4}$
Earplug	$1.08(1) \times 10^{-4}$	$1.8^{+3.2}_{-0.8} \times 10^{-4}$
Sphere of similar dimensions as earplug	$6.34(8) \times 10^{-5}$	-

TABLE II. Table of Results for c

Tab. II lists the obtained values of c together with the previously calculated expected values. In line with our expectations that larger objects undergo more drag, we observe that the value of c is larger for the ping pong ball. Most importantly, both of the obtained values of c are within uncertainty, which supports the veracity of our earlier calculations. This provides limited evidence that projectile shape does affect drag, because from here we can deduce that the earplug seems to experience slightly more drag than if it were a sphere of similar dimensions. Thus our data supports the contention that shape and size do affect drag.

V. CONCLUSION

In conclusion, we have investigated the effects of drag on different projectiles traveling in two dimensions. We were able to fit our theoretical model that includes drag to the experimental trajectory. Using two projectiles which were of different shapes and sizes, we obtained experimental values of the drag coefficient c which are in agreement with our expectations. Thus, the experiment has succeeded in providing some preliminary evidence that shape and size do matter in determining drag.

However, despite this current result, there are still many possibilities to improve our experiment in order to further investigate this matter. The most apparent would be to experiment with a greater variety of projectiles of different shapes and sizes. Another interesting prospect is to take forces other than gravity and drag into account, namely lift. Lift is defined as any non-gravitational force acting on the projectile which is perpendicular to its velocity (drag being the force acting antiparallel to the velocity). Lift for a spherical or cylindrical projectile is most likely caused by the Magnus force, which is determined by the projectile's spin.

The Magnus force is the result of a spinning projectile that causes a difference in air pressure between the two opposing sides of the projectile, giving rise to a force perpendicular to the projectile's translational velocity. The direction of the force depends on the direction of the spin. In our case, a small amount of spin is imparted to the projectile when the experimenter lobs it into the air. The spin is in the forward direction, thus causing a force in the downward direction. We estimate that during its $\sim$1 second journey, each projectile undergoes no more than 10 full rotations. According to Nathan [6], for a spherical projectile, the strength of the Magnus force F_M can be estimated using the expression

$$F_M = 2\kappa\rho r\omega A v, \tag{13}$$

where κ is the shape coefficient from Eq. 11, ρ is the density of the fluid, r is the radius of the sphere, ω is the angular frequency, A is the cross-sectional area of the sphere, and v is the translational velocity. For our case, $\omega < 60$, and $v \sim 1$ m/s. Thus for the ping pong ball, $F_M < 0.005$ N. In comparison, the ball experiences a force of ~ 0.03 N due to gravity. This means that the Magnus force is indeed small but not negligible, and so it presents a good avenue to further refine our model.

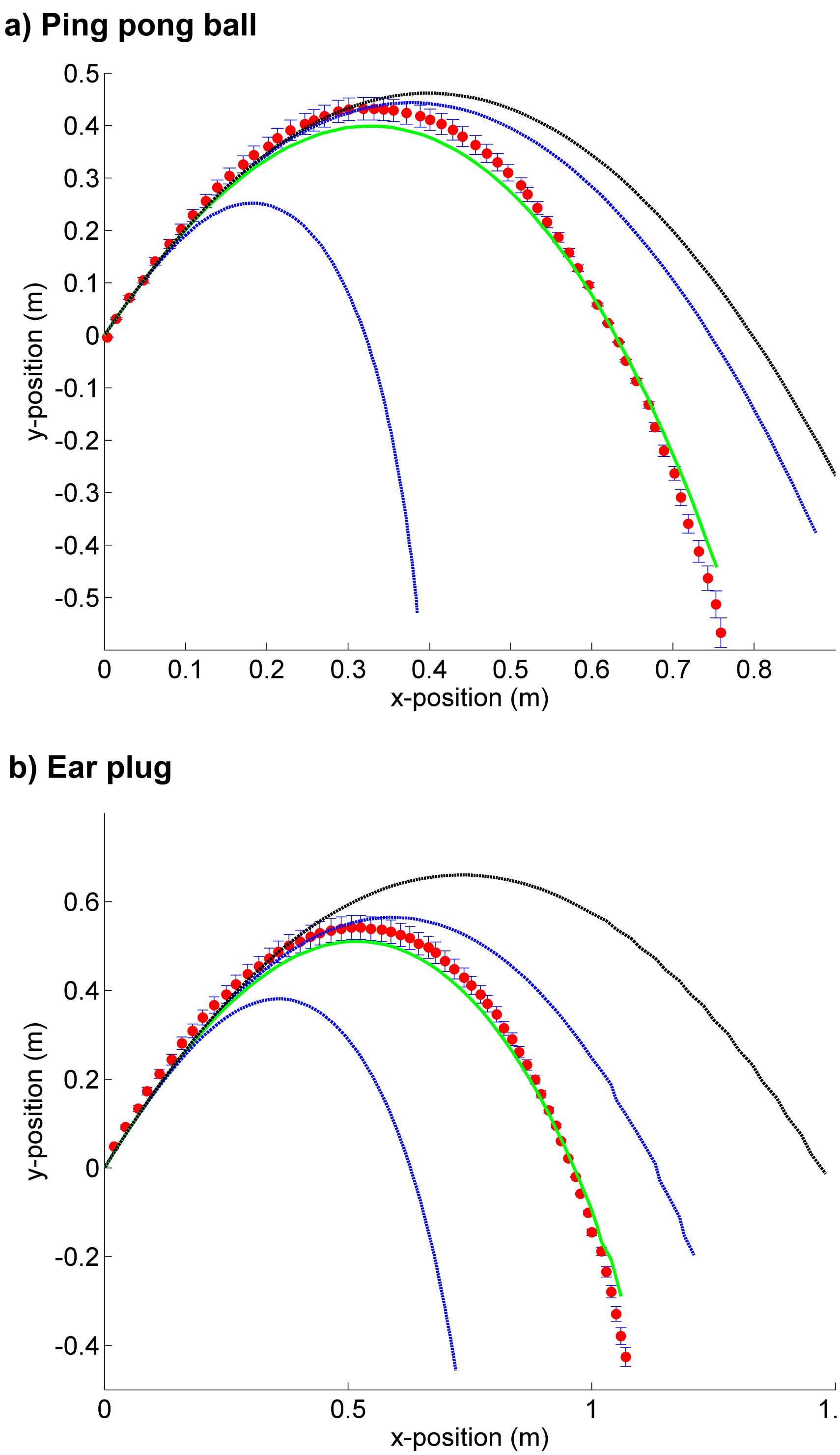

FIG. 3. Experimental results for a) ping pong ball and b) ear plug. The red dots plot the actual trajectory of the projectile. The blue lines are plotted using the minimum and maximum values of c. The green line is plotted using a c which is the average of these two, and is regarded as the obtained value for c. The black dashed line is the trajectory calculated without taking drag into account.

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